

Full box spaces of free groups

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Abstract

In this paper we investigate full box spaces and coarse equivalences between them. We do this in two parts. In part one we compare the full box spaces of free groups on different numbers of generators. In particular the full box space of a free group F_k is not coarsely equivalent to the full box space of a free group F_d , if $d \geq 8k + 10$. In part two we compare $\square_f \mathbb{Z}^n$ to the full box spaces of 2-generated groups. In particular we prove that the full box space of \mathbb{Z}^n is not coarsely equivalent to the full box space of any 2-generated group, if $n \geq 3$.

1 Introduction

Given a finitely generated group G we can consider a collection of finite index normal subgroups $(N_i)_i$, and we can create the metrized disjoint union of all G/N_i . The metric of $\coprod G/N_i$ is defined as follows: $d(x, y) = d_{G/N_i}(x, y)$ if $x, y \in G/N_i$ and $d(x, y) = \text{diam}(G/N_i) + \text{diam}(G/N_j)$ if $x \in G/N_i$ and $y \in G/N_j$ with $i \neq j$. Note that G/N_i is endowed with the word metric coming from a fixed finite generating set S in G .

If $(N_i)_i$ is a decreasing sequence of finite index normal subgroups of G with trivial intersection, then the metrized disjoint union $\coprod G/N_i$ is called a box space of G , denoted by $\square_{N_i} G$. Note that this only exists if G is residually finite.

In this paper we will mainly be concerned with the full box space $\square_f G$ of G , which is the metrized disjoint union of all finite quotients of G . We will study these spaces up to coarse equivalence:

Definition 1.1. *Let (X, d_X) and (Y, d_Y) be metric spaces. Then a map $f: X \rightarrow Y$ is a coarse equivalence if $f(X)$ is C -dense in Y for some constant C and*

$$d_X(x_n, y_n) \rightarrow +\infty \iff d_Y(f(x_n), f(y_n)) \rightarrow +\infty$$

for any two sequences $(x_n)_n$ and $(y_n)_n$ in X .

An interesting way of thinking about a coarse equivalence between box spaces follows from Lemma 1 of [KV15].

Lemma 1.2 (Khukhro, Valette). *Let $\Phi: \square_{N_i} G \rightarrow \square_{M_i} H$ be a coarse equivalence between the box spaces of the residually finite finitely generated groups G and H . Then there exists a constant A and an almost permutation ϕ between the components of $\square_{N_i} G$ and the components of $\square_{M_i} H$ such that $\Phi|_{G/N_i}$ is an (A, A) -quasi-isometry between G/N_i and $\phi(G/N_i)$.*

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Note that an almost permutation between sets A and B is a bijection between a co-finite subset of A and a co-finite subset of B .

Since G/N_i and $\phi(G/N_i)$ are (A, A) -quasi-isometric we have that both the diameter and the order of G/N_i and $\phi(G/N_i)$ are quite similar. In Section 2 we will compare the full box spaces of the free groups, using the similarity of the order, i.e. we will compare the normal subgroup growth of different free groups.

Theorem 1.3. *Let $2 \leq k \leq d$ with $2(k+1) < \frac{(d-1)^2}{4d}$. Then $\square_f F_d$ is not coarsely equivalent to $\square_f F_k$.*

In Section 3 we will use the similarity of diameter to prove that $\square_f \mathbb{Z}^n$ is not coarsely equivalent to the full box space of a 2-generated group.

Theorem 1.4. *Let $n \geq 3$ and let H be a 2-generated group. Then $\square_f H$ is not coarsely equivalent to $\square_f \mathbb{Z}^n$.*

The most notable thing about the proofs of these theorems is that we do not show that the components of the box spaces are different, we show that one of the box spaces has too many small components compared to the other. In other words, it is still open whether for every box space $\square_{N_i} F_d$, there exists a box space $\square_{M_i} F_k$, which is coarsely equivalent to $\square_{N_i} F_d$.

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2 The full box spaces of the free groups

In this section we will prove that the full box spaces of free groups are different, at least if the amount of generators is sufficiently different. This suggests that the full box spaces of all free groups are different.

In the proof we make use of normal subgroup growth, for further reading on (normal) subgroup growth we refer to [LS12]. For this paper we only need $a_n^\triangleleft(G)$, which is the amount of normal subgroups of the group G of index n .

Proof of Theorem 1.3. Suppose that the full box spaces of the free groups F_d and F_k are coarsely equivalent where $2(k+1) < \frac{(d-1)^2}{4d}$, i.e. there is a coarse equivalence Φ between $\square_f F_d$ and $\square_f F_k$. Due to Lemma 1.2 there is an almost permutation ϕ between the components of $\square_f F_d$ and the components of $\square_f F_k$. As there is some C' such that $\text{Im } \Phi$ is C' -dense, components of order less than some n must be mapped to a component of order less than $n \cdot |B[0, C']|$, where $B[0, C']$ is the closed ball of radius C' . Now set $C = |B[0, C']|$ and set D equal to the number of components that are not in the domain of ϕ .

So $|\{N \triangleleft F_d \mid \#(F_d/N) \leq n\}| - D$ is not greater than $|\{N \triangleleft F_k \mid \#(F_k/N) \leq Cn\}|$ for any n . Note that

$$\{N \triangleleft F_d \mid \#(F_d/N) \leq n\} = \{N \triangleleft F_d \mid [F_d : N] \leq n\} = \sum_{i=1}^n a_i^\triangleleft(F_d),$$

so we find the following inequality:

$$\sum_{i=1}^{Cn} a_i^\triangleleft(F_k) + D \geq \sum_{i=1}^n a_i^\triangleleft(F_d) \geq a_n^\triangleleft(F_d)$$

It suffices to find an n for which this is not the case.

Let n be a power of 2, $n = 2^m$. Then $a_n^\triangleleft(F_d) \geq 2^{cm^2}$ if $c < \frac{(d-1)^2}{4d}$ due to Theorem 3.7 of [LS12].

As $\frac{(d-1)^2}{4d} > 2(k+1)$ we can take $c = 2(k+1) + 2\delta$, where $\delta > 0$. Due to Theorem 2.6 and Lemma 2.5 of [LS12], $a_i^{\triangleleft}(F_k) \leq i^k i^{2(k+1)\log_2(i)}$ for every $i \in \mathbb{N}$. By combining these two bounds we can make the following computation:

$$\begin{aligned}
2^{cm^2} - D &\leq a_n^{\triangleleft}(F_d) - D \\
&\leq \sum_{i=1}^{Cn} a_i^{\triangleleft}(F_k) \\
&\leq \sum_{i=1}^{Cn} i^k i^{2(k+1)\log_2(i)} \\
&\leq \sum_{i=1}^{Cn} (Cn)^k (Cn)^{2(k+1)\log_2(Cn)} \\
&= C^{k+1} n^{k+1} C^{2(k+1)(\log_2(n)+\log_2(C))} n^{2(k+1)(\log_2(n)+\log_2(C))} \\
&= 2^{(k+1)\log_2(C)} 2^{m(k+1)} 2^{2(k+1)(m+\log_2(C))\log_2(C)} 2^{2m(k+1)(m+\log_2(C))} \\
&= 2^{(k+1)\log_2(C)+m(k+1)+2(k+1)(m+\log_2(C))\log_2(C)+2m(k+1)(m+\log_2(C))} \\
&= 2^{2(k+1)m^2+m(k+1)+4m(k+1)\log_2(C)+2(k+1)\log_2(C)^2+(k+1)\log_2(C)} \\
&= 2^{(k+1)(2m^2+m+4m\log_2(C)+2\log_2(C)^2+\log_2(C))}
\end{aligned}$$

Now we can take $m \gg 0$ such that $2m^2 + m + 4m\log_2(C) + 2\log_2(C)^2 + \log_2(C) \leq (2 + \frac{\delta}{k+1})m^2$ and $D < 2^{cm^2} - 2^{(2(k+1)+\delta)m^2}$. But then we find the following contradiction.

$$\begin{aligned}
2^{cm^2} - D &\leq 2^{(k+1)(2m^2+m+4m\log_2(C)+2\log_2(C)^2+\log_2(C))} \\
&\leq 2^{(k+1)(2+\frac{\delta}{k+1})m^2} \\
&= 2^{(2(k+1)+\delta)m^2} \\
&< 2^{cm^2} - D
\end{aligned}$$

This proves that $\square_f F_d$ is not coarsely equivalent to $\square_f F_k$ for $2(k+1) < \frac{(d-1)^2}{4d}$. \square

To use Theorem 1.3 we only need to find appropriate values for k and d . The condition $2(k+1) < \frac{(d-1)^2}{4d}$ is satisfied if and only if d is not smaller than $8k+10$. For example $\square_f F_2$ is not coarsely equivalent to $\square_f F_{26}$.

3 The full box spaces of \mathbb{Z}^n

In this section we will prove that the full box space of \mathbb{Z}^n is not coarsely equivalent to the full box space of a 2-generated group for every $n \geq 3$. To do so we will compare the growth in k of $\#\{\text{quotients with diameter} \leq k\}$, which we will call the diameter growth of the components of these full box spaces. Note that the term diameter growth is often used to compare the growth of the diameter with that of the index. Once we know the diameter growth we can compare the full box spaces using the following result:

Proposition 3.1. *Let G and H be two groups, with $\square_f G$ coarsely equivalent to $\square_f H$ and let $a \in \mathbb{N}$. If $\#\{N \triangleleft G \mid \text{diam}(G/N) \leq k\} = \mathcal{O}(k^a)$, then $\#\{N \triangleleft H \mid \text{diam}(H/N) \leq k\} = \mathcal{O}(k^a)$.*

Proof. As there exists a coarse equivalence $\Phi: \square_f G \rightarrow \square_f H$ we can use Lemma 1.2 to find an almost permutation ϕ between the components of $\square_f G$ and the components of $\square_f H$ such that $\Phi|_{G/N}$ is an (A, A) -quasi-isometry between G/N and $\phi(G/N)$, if G/N lies in the domain of ϕ . Therefore $\text{diam}(\phi(G/N)) \leq A \text{diam}(G/N) + A$.

We can take a constant C such that $\#\{N \triangleleft G \mid \text{diam}(G/N) \leq k\} \leq Ck^a$ for every k . Now ϕ is an almost permutation, so we can define $D = |(\text{Im } \phi)^c|$. Then we can bound $\#\{N \triangleleft H \mid \text{diam}(H/N) \leq k\}$ as follows:

$$\begin{aligned} \#\{N \triangleleft H \mid \text{diam}(H/N) \leq k\} &\leq \#\{N \triangleleft H \mid \text{diam}(H/N) \leq k, H/N \in \text{Im}(\phi)\} + D \\ &\leq \#\{N \triangleleft G \mid \text{diam}(\phi(G/N)) \leq k, G/N \in \text{dom}(\phi)\} + D \\ &\leq \#\{N \triangleleft G \mid \text{diam}(G/N) \leq Ak + A^2, G/N \in \text{dom}(\phi)\} + D \\ &\leq \#\{N \triangleleft G \mid \text{diam}(G/N) \leq Ak + A^2\} + D \\ &\leq C(Ak + A^2)^a + D. \end{aligned}$$

So $\#\{N \triangleleft H \mid \text{diam}(H/N) \leq k\} = \mathcal{O}(k^a)$. □

Now we want to calculate the diameter growth of $\square_f \mathbb{Z}^n$.

Proposition 3.2. *For every $n \in \mathbb{N}$ we have*

$$\#\{N \triangleleft \mathbb{Z}^n \mid \text{diam}(\mathbb{Z}^n/N) \leq k\} = \Omega(k^{n^2}).$$

Proof. Fix a k and consider the subgroups of \mathbb{Z}^n generated by x_1, \dots, x_n with $\frac{k}{2n} < x_{ii} \leq \frac{k}{n}$ and $|x_{ij}| \leq \frac{k}{2n^2}$ for every $i \neq j$, where $x_i = (x_{i1}, \dots, x_{in})$. The number of possibilities for x_1, \dots, x_n is $(\frac{k}{2n})^n (\frac{2k}{2n^2} + 1)^{n(n-1)}$. This is more than $\frac{1}{(2n)^n} \frac{1}{n^{2n(n-1)}} k^{n^2}$. So it suffices to show that all these subgroups N are different and the diameter of \mathbb{Z}^n/N is not greater than k .

To show that these subgroups are different take $N = N'$ where N is generated by x_1, \dots, x_n and N' is generated by x'_1, \dots, x'_n . For every $i \leq n$ we can take $x'_i = a_1 x_1 + \dots + a_n x_n$ with $a_1, \dots, a_n \in \mathbb{Z}$, since $N = N'$. Now take $j \neq i$ such that a_j is maximal. By projecting on the j^{th} -component we get the following:

$$\begin{aligned} \frac{k}{2n^2} &\geq |a_1 x_{1j} + \dots + a_n x_{nj}| \\ &\geq |a_j| x_{jj} - \sum_{k \neq j} |a_k x_{kj}| \\ &> \frac{k}{2n} |a_j| - \sum_{k \neq j} \frac{k}{2n^2} |a_k| \\ &\geq \frac{k}{2n} |a_j| - (n-1) \frac{k}{2n^2} |a_j| \\ &= \frac{k}{2n^2} |a_j|. \end{aligned}$$

We can conclude that $a_j = 0$, therefore only a_i can be different from 0, which has to be equal to 1, because $\frac{k}{2n} < x_{ii}, x'_{ii} \leq \frac{k}{n}$, so $x'_i = x_i$. This is true for every i , so N and N' are generated by the same vectors x_1, \dots, x_n .

To prove that $\text{diam}(\mathbb{Z}^n/N) \leq k$ suppose there is such a subgroup N for which $\text{diam}(\mathbb{Z}^n/N) > k$. So there is an element in \mathbb{Z}^n/N such that for every representing vector $y = (y_1, \dots, y_n)$ in \mathbb{Z}^n

we have $\sum_{i=1}^n |y_i| > k$. Let y be the representing vector for which $\|y\|$ is minimal and let i be such that $|y_i|$ is maximal. Without loss of generality we may assume y_i to be positive. Now as $\sum_{i=1}^n |y_i| > k$, we find that $y_i > \frac{k}{n} \geq x_{ii} > 0$ and we get the following:

$$\begin{aligned}
\|y - x_i\| &= \sum_{j=1}^n |y_j - x_{ij}| \\
&\leq y_i - x_{ii} + \sum_{j \neq i} (|y_j| + |x_{ij}|) \\
&< y_i - \frac{k}{2n} + \sum_{j \neq i} \left(|y_j| + \frac{k}{2n^2} \right) \\
&= \|y\| - \frac{k}{2n^2}.
\end{aligned}$$

Now $y - x_i$ is a smaller representing vector of the same element as y , which is a contradiction. So for all these subgroups N we have $\text{diam}(\mathbb{Z}^n/N) \leq k$, which proves that $\#\{N \triangleleft \mathbb{Z}^n \mid \text{diam}(\mathbb{Z}^n/N) \leq k\} = \Omega(k^{n^2})$. \square

In the proof of Theorem 1.4 it will suffice to know the diameter growth of a 2-generated virtually \mathbb{Z}^n group H . Consequently H must be \mathbb{Z}^n -by-finite, because one can turn the finite index subgroup \mathbb{Z}^n into a finite index normal subgroup by taking the intersection of all $g^{-1}\mathbb{Z}^n g$, which is again \mathbb{Z}^n as it is a finite index subgroup in \mathbb{Z}^n . In order to calculate this growth we will first restrict the normal subgroups of H to the finite index normal subgroup $\mathbb{Z}^n \triangleleft H$. To better understand these normal subgroups of \mathbb{Z}^n we define minimal generating sets.

Definition 3.3. A minimal generating set of $N \triangleleft \mathbb{Z}^n$ is the subset $\{x_1, \dots, x_n\}$ of N where x_1 is the smallest vector in N (for the euclidean norm) and x_i is the smallest vector in $N \setminus \langle x_1, \dots, x_{i-1} \rangle$ such that $N \cap \text{span}(x_1, \dots, x_i) = \langle x_1, \dots, x_i \rangle$.

A minimal generating set is a generating set of N , because it is linearly independent and therefore $N = N \cap \text{span}(x_1, \dots, x_n) = \langle x_1, \dots, x_n \rangle$. Note that such a generating set always exists. Also note that a subset of a minimal generating set is a minimal generating set of what it generates. This notion will be important to control the diameter of $\mathbb{Z}^n/(N \cap \mathbb{Z}^n)$.

Lemma 3.4. For every $n \in \mathbb{N}$ there exists a constant $D_n \in \mathbb{N}$ such that for every subgroup N of \mathbb{Z}^n and every minimal generating set $\{x_1, \dots, x_n\}$ we have $\|a_1 x_1 + \dots + a_n x_n\| \geq \frac{1}{D_n} \max_i \|a_i x_i\|$ for every $a_1, \dots, a_n \in \mathbb{R}$.

As we will do in the proof of Lemma 3.4, we define D_n recursively with $D_1 = 1$ and $D_n = D_{n-1}^2 (4n^2 D_{n-1}^3)^n$.

If the minimal generating set we choose happens to be orthogonal, this lemma would be obvious. The main idea behind the proof is to show that minimal generating sets are sufficiently similar to being orthogonal. In the proof we will assume that Lemma 3.4 is true up to some value n . We will use this to prove an intermediate result (Lemma 3.5 for $m = n$) and then we will use that to show that Lemma 3.4 is true for $n + 1$.

Lemma 3.5. *Let $\{x_1, \dots, x_{m+1}\}$ be a minimal generating set and let p be the orthogonal projection on $\text{span}(x_1, \dots, x_m)$, so we can write $p(x_{m+1}) = a_1x_1 + \dots + a_mx_m$. Suppose Lemma 3.4 is satisfied for all $n \leq m$. Then $|a_m| \leq \frac{m}{2}D_m$ and $|a_i| \leq \frac{m}{2}D_m^2$ for all $i < m$.*

For this lemma we will also assume that $D_{i+1} \geq \frac{i}{2}D_i^2 + 1$ for every $i \geq 1$, which will be the case in the proof of Lemma 3.4.

Proof. We proceed by contradiction. Let $\{x_1, \dots, x_{m+1}\}$ be a minimal generating set with the smallest m such that it does not satisfy Lemma 3.5. Then we find

$$\|p(x_{m+1})\| = \|a_1x_1 + \dots + a_mx_m\| \geq \frac{1}{D_m} \max_i \|a_ix_i\| \geq \frac{|a_m|}{D_m} \|x_m\|.$$

However as $\{x_1, \dots, x_{m+1}\}$ is a minimal generating set we have that for every b_1, \dots, b_m in \mathbb{Z} $\|x_{m+1}\| \leq \|x_{m+1} - b_1x_1 - \dots - b_mx_m\|$, we even have $\|p(x_{m+1})\| \leq \|p(x_{m+1}) - b_1x_1 - \dots - b_mx_m\|$, because the projections of both vectors onto $\text{span}(x_1, \dots, x_m)^\perp$ are equal. If we take b_i such that $|b_i - a_i| \leq \frac{1}{2}$, then we find the following inequality:

$$\begin{aligned} \|p(x_{m+1})\| &\leq \|p(x_{m+1}) - b_1x_1 - \dots - b_mx_m\| \\ &\leq \|(a_1 - b_1)x_1\| + \dots + \|(a_m - b_m)x_m\| \leq \frac{m}{2} \|x_m\|. \end{aligned}$$

Combining these inequalities we conclude that $|a_m| \leq \frac{m}{2}D_m$. As we assume this minimal generating set does not satisfy the lemma there must be an a_i such that $|a_i| > \frac{m}{2}D_m^2$, let l be the largest such i .

Now let p_i be the orthogonal projection onto $\text{span}(x_1, \dots, x_i)$. We will use these projections to bound the corresponding $|a_i|$. We already have $p(x_{m+1}) = a_1x_1 + \dots + a_mx_m$. Now we take something similar for the projections p_i :

$$\begin{aligned} p_{m-1}(a_mx_m) &= a_{m-1,m}x_{m-1} + \dots + a_{1,m}x_1 \\ p_{m-2}((a_{m-1} + a_{m-1,m})x_{m-1}) &= a_{m-2,m-1}x_{m-2} + \dots + a_{1,m-1}x_1 \\ &\vdots \\ p_l((a_{l+1} + a_{l+1,m} + \dots + a_{l+1,l+2})x_{l+1}) &= a_{l,l+1}x_l + \dots + a_{1,l+1}x_1 \end{aligned}$$

Let m' be such that $l \leq m' < m$. As before we have

$$\begin{aligned} \|p_{m'}(x_{m+1})\| &= \|a_1x_1 + \dots + a_{m'}x_{m'} + a_{1,m}x_1 + \dots + a_{m',m}x_{m'} + \dots + a_{m',m'+1}x_{m'}\| \\ &\geq \frac{1}{D_{m'}} \|(a_{m'} + a_{m',m} + \dots + a_{m',m'+1})x_{m'}\| \geq \frac{|a_{m'} + a_{m',m} + \dots + a_{m',m'+1}|}{D_{m'}} \|x_{m'}\|. \end{aligned}$$

As before we can take b_1, \dots, b_m in \mathbb{Z} such that $|b_i - a_i - a_{i,m} - \dots - a_{i,m'+1}| \leq \frac{1}{2}$ for every i . Now we find

$$\|p_{m'}(x_{m+1})\| \leq \|p_{m'}(x_{m+1}) - b_1x_1 - \dots - b_{m'}x_{m'}\| \leq \frac{1}{2} \|x_1\| + \dots + \frac{1}{2} \|x_{m'}\| \leq \frac{m'}{2} \|x_{m'}\|.$$

So $\frac{m'}{2}D_{m'} \geq |a_{m'} + a_{m',m} + \dots + a_{m',m'+1}|$. As m is assumed to be the smallest value for which this lemma is not true, we have that when $p_{m'}(x_{m'+1})$ is written as a linear combination of

$x_1, \dots, x_{m'}$, where the coefficient of x_m is not greater than $\frac{m'}{2}D_{m'}$ and the other coefficients are not greater than $\frac{m'}{2}D_{m'}^2$. Now as

$$p_{m'}((a_{m'+1} + a_{m'+1,m} + \dots + a_{m'+1,m'+2})x_{m'+1}) = a_{m',m'+1}x_{m'} + \dots + a_{1,m'+1}x_1$$

we have $|a_{m',m'+1}| \leq \frac{m'}{2}D_{m'}|a_{m'+1} + a_{m'+1,m} + \dots + a_{m'+1,m'+2}| \leq \frac{m'}{2}D_{m'}\frac{m'+1}{2}D_{m'+1}$ and $|a_{i,m'+1}| \leq \frac{m'}{2}D_{m'}^2|a_{m'+1} + a_{m'+1,m} + \dots + a_{m'+1,m'+2}| \leq \frac{m'}{2}D_{m'}^2\frac{m'+1}{2}D_{m'+1}$ for $i < m'$. Now we had $\frac{l}{2}D_l \geq |a_l + a_{l,m} + \dots + a_{l,m'+1}|$, so using the fact that $D_{i+1} \geq \frac{i}{2}D_i^2 + 1$ and $iD_i \leq (i+1)D_{i+1}$ for every $i \geq 1$, we can make the following computation.

$$\begin{aligned} |a_l| &\leq |a_{l,m}| + \dots + |a_{l,l+1}| + \frac{l}{2}D_l \\ &\leq \frac{m-1}{2}D_{m-1}^2\frac{m}{2}D_m + \dots + \frac{l+1}{2}D_{l+1}^2\frac{l+2}{2}D_{l+2} + \frac{l}{2}D_l\frac{l+1}{2}D_{l+1} + \frac{l}{2}D_l \\ &\leq \frac{m-1}{2}D_{m-1}^2\frac{m}{2}D_m + \dots + \frac{l+1}{2}D_{l+1}^2\frac{l+2}{2}D_{l+2} + \frac{l+1}{2}D_{l+1}\left(\frac{l}{2}D_l + 1\right) \\ &\leq \frac{m-1}{2}D_{m-1}^2\frac{m}{2}D_m + \dots + \frac{l+1}{2}D_{l+1}^2\frac{l+2}{2}D_{l+2} + \frac{l+1}{2}D_{l+1}^2 \\ &\leq \frac{m-1}{2}D_{m-1}^2\frac{m}{2}D_m + \dots + \frac{l+1}{2}D_{l+1}^2\frac{l+2}{2}D_{l+2} + \frac{l+2}{2}D_{l+2} \\ &\vdots \\ &\leq \frac{m-1}{2}D_{m-1}^2\frac{m}{2}D_m + \frac{m}{2}D_m \\ &\leq \left(\frac{m-1}{2}D_{m-1}^2 + 1\right)\frac{m}{2}D_m \\ &\leq \frac{m}{2}D_m^2 \end{aligned}$$

But we assumed $|a_l| > \frac{m}{2}D_m^2$, and so we have a contradiction, which proves this lemma. \square

Now we can use this result to prove Lemma 3.4.

Proof of Lemma 3.4. We define D_n recursively with $D_1 = 1$ and $D_n = D_{n-1}^2 (4n^2 D_{n-1}^3)^n$. For every subgroup $N \triangleleft \mathbb{Z}^n$ we can take a minimal generating set x_1, \dots, x_n .

Let n be the smallest value for which the lemma is not true, i.e. there exist a_i such that $\frac{1}{D_n} \max_i \{\|a_i x_i\|\} > \|a_1 x_1 + \dots + a_n x_n\|$. As the lemma is obvious for $n = 1$, we may assume that $n \geq 2$.

First we observe that $\|a_i x_i\|$ must be similar for all i , that is $\min_i \{\|a_i x_i\|\} > \frac{1}{2D_{n-1}} \max_i \{\|a_i x_i\|\}$. We can see this by combining the reverse triangular inequality with the fact that a subset of a minimal generating set is a minimal generating set of what it generates: $\frac{1}{D_n} \max_i \{\|a_i x_i\|\} > \|a_1 x_1 + \dots + a_n x_n\| \geq \frac{1}{D_{n-1}} \max_i \{\|a_i x_i\|\} - \min_i \{\|a_i x_i\|\}$. So we get the desired result that $\min_i \{\|a_i x_i\|\} > \left(\frac{1}{D_{n-1}} - \frac{1}{D_n}\right) \max_i \{\|a_i x_i\|\} \geq \frac{1}{2D_{n-1}} \max_i \{\|a_i x_i\|\}$.

To continue we would prefer for x_n to be orthogonal to $\text{span}(x_1, \dots, x_{n-1})$. However a partial result will suffice. We will show that the angle between x_n and the span of x_1, \dots, x_{n-1} can not be arbitrarily small, which will prove the lemma. So let p be the orthogonal projection onto $\text{span}(x_1, \dots, x_{n-1})$.

Now distinguish two cases according to whether or not $nD_{n-1}^2|a_n|$ is greater or smaller than $\max_i\{|a_i|\}$.

Suppose $\max_i\{|a_i|\} > nD_{n-1}^2|a_n|$. As such we can write $p(x_n)$ as the linear combination $a'_1x_1 + \dots + a'_{n-1}x_{n-1}$.

Due to Lemma 3.5 we know that $|a'_i| \leq \frac{n}{2}D_{n-1}^2$ for every i . Now we can take k such that $|a_k|$ is maximized. By combining $|a'_k| \leq \frac{n}{2}D_{n-1}^2$ with $\max_i\{|a_i|\} = |a_k| > nD_{n-1}^2|a_n|$ we find that

$|a_k + a'_ka_n| \geq |a_k| - \frac{nD_{n-1}^2}{2}|a_n| \geq \frac{1}{2}|a_k|$. This admits the following computation:

$$\begin{aligned}
\frac{1}{D_n} \max_i\{\|a_ix_i\|\} &\geq \|a_1x_1 + \dots + a_nx_n\| \\
&\geq \|p(a_1x_1 + \dots + a_{n-1}x_{n-1} + a_nx_n)\| \\
&\geq \|a_1x_1 + \dots + a_{n-1}x_{n-1} + a_np(x_n)\| \\
&\geq \|(a_1 + a'_1a_n)x_1 + \dots + (a_{n-1} + a'_{n-1}a_n)x_{n-1}\| \\
&\geq \frac{1}{D_{n-1}} \max_i\{\|(a_i + a'_ia_n)x_i\|\} \\
&\geq \frac{1}{2D_{n-1}} \|a_kx_k\| \\
&\geq \frac{1}{2D_{n-1}} \min_i\{\|a_ix_i\|\} \\
&\geq \frac{1}{4D_{n-1}^2} \max_i\{\|a_ix_i\|\}.
\end{aligned}$$

Now $n \geq 2$, so $D_n = 2D_{n-1}(2n^2D_{n-1})^n > 4D_{n-1}^2$, which contradicts the earlier computations.

Up to this point we essentially only used that x_n can not be shortened by adding a linear combination $\lambda_1x_1 + \dots + \lambda_{n-1}x_{n-1}$ with $\lambda_1, \dots, \lambda_{n-1} \in \mathbb{Z}$. However if $\max_i\{|a_i|\} \leq nD_{n-1}^2|a_n|$ this will not be possible. For example for every $\varepsilon > 0$ we have $(2, 0, 0, 0, 0)$, $(0, 2, 0, 0, 0)$, $(0, 0, 2, 0, 0)$, $(0, 0, 0, 2, 0)$, $(1, 1, 1, 1, \varepsilon)$, but the group generated by these vectors contains $(0, 0, 0, 0, 2\varepsilon)$. In the continuation of this proof we will look for a vector like $(0, 0, 0, 0, 2\varepsilon)$, more precisely a short vector that is almost orthogonal to x_1, \dots, x_{n-1} .

As $\max_i\{|a_i|\} \leq nD_{n-1}^2|a_n|$, we have $nD_{n-1}^2\|a_nx_n\| \geq \max_i\|a_ix_i\|$, as x_n is the biggest vector in the basis $\{x_1, \dots, x_n\}$. Let e be a unit vector perpendicular to $\text{span}(x_1, \dots, x_{n-1})$. Then $\frac{nD_{n-1}^2}{D_n}\|a_nx_n\| \geq \|a_1x_1 + \dots + a_nx_n\| \geq |a_nx_n \cdot e|$, so $\|x_n\| \geq \frac{D_n}{nD_{n-1}^2}|x_n \cdot e|$.

Now for every $m \in \mathbb{N}$ we can take $p(mx_n) = b_1x_1 + \dots + b_{n-1}x_{n-1} + c_1x_1 + \dots + c_{n-1}x_{n-1}$ with $b_i \in \mathbb{Z}$ and $|c_i| \leq \frac{1}{2}$ for every i . What we are looking for is an m such that c_1, \dots, c_{n-1} are close to zero. In that case $p(mx_n - b_1x_1 - \dots - b_{n-1}x_{n-1})$ is small.

To make this precise: for every $i < n$ there exists a $k_i \in \mathbb{N}$ such that $c_i \in \left[\frac{k_i}{4n^2D_{n-1}^3}, \frac{k_i+1}{4n^2D_{n-1}^3}\right]$, with k_i between $-2n^2D_{n-1}^3$ and $2n^2D_{n-1}^3 - 1$. Now due to the pigeonhole principle there will be an $m, m' \leq (4n^2D_{n-1}^3)^{n-1}$ with $k_i = k'_i$ for every i . Now $(m - m')x_n$ will be the vector we are looking for, because $c'_i - c_i \in \left[\frac{-1}{4n^2D_{n-1}^3}, \frac{1}{4n^2D_{n-1}^3}\right]$. As x_1 is the smallest vector in N we can

make the following computation:

$$\begin{aligned}
\|x_1\|^2 &\leq \|(b_1 - b'_1)x_1 + \dots + (b_{n-1} - b'_{n-1})x_{n-1} + (m' - m)x_n\|^2 \\
&= \|(b_1 - b'_1)x_1 + \dots + (b_{n-1} - b'_{n-1})x_{n-1} + p(m'x_n) - p(mx_n)\|^2 + |m' - m|^2 |x_n \cdot e|^2 \\
&\leq \left(\sum_{i=1}^{n-1} \|(c'_i - c_i)x_i\| \right)^2 + (4n^2 D_{n-1}^3)^{2n-2} \frac{n^2 D_{n-1}^4}{D_n^2} \|x_n\|^2 \\
&\leq \left(\sum_{i=1}^{n-1} \frac{\|x_i\|}{4n^2 D_{n-1}^3} \right)^2 + \left(\frac{(4n^2 D_{n-1}^3)^n}{4n D_{n-1} D_n} \right)^2 \|x_n\|^2 \\
&\leq \left(\frac{(n-1) \|x_n\|}{4n^2 D_{n-1}^3} \right)^2 + \frac{1}{8n^2 D_{n-1}^6} \|x_n\|^2 \\
&< \frac{1}{4n^2 D_{n-1}^6} \|x_n\|^2
\end{aligned}$$

However, this contradicts the earlier results that $\max_i |a_i| \leq 2n D_{n-1}^2$ and $\min_i \|a_i x_i\| \geq \|a_n x_n\|$

$$\|x_1\| = \frac{1}{|a_1|} \|a_1 x_1\| \geq \frac{1}{n D_{n-1}^2 |a_n|} \min_i \|a_i x_i\| \geq \frac{1}{2n D_{n-1}^3} \frac{1}{|a_n|} \|a_n x_n\| \geq \frac{1}{2n D_{n-1}^3} \|x_n\|.$$

So for every $a_1, \dots, a_n \in \mathbb{R}$ we have $\|a_1 x_1 + \dots + a_n x_n\| \geq \frac{1}{D_n} \max_i \{\|a_i x_i\|\}$. \square

As mentioned earlier this lemma will help us to control the diameter of $\mathbb{Z}^n/(N \cap \mathbb{Z}^n)$. However we need to control the diameter of H/N . We will show that H must be \mathbb{Z}^n -by-finite and then we will consider $\mathbb{Z}^n/(N \cap \mathbb{Z}^n)$. While it is not necessarily true that $\text{diam}(H/N) \geq \text{diam}(\mathbb{Z}^n/(N \cap \mathbb{Z}^n))$, it is true up to a constant.

Lemma 3.6. *Let G and H be two groups such that H is G -by-finite. Then there exists a constant C such that $C \text{diam}(H/N) \geq \text{diam}(G/(N \cap G))$ for every $N \triangleleft H$.*

Proof. Due to Proposition 2 of [Khu12], there exists a C' such that $\text{diam}_G(G/(N \cap G)) \leq C' \text{diam}_H(G/(N \cap G))$ for every $N \in H$. So it suffices to show that there exists a C such that $\text{diam}_H(G/(N \cap G)) \leq C \text{diam}(H/N)$. As H is G -by-finite we can take $F = H/G$ finite and set $C = 3|F|$.

We can take $g \in G$ such that $|g|_H = |gN|_{H/N} = \text{diam}_H(G/(N \cap G))$. Now take a path between 1 and g and take $1 = b_0, b_1, \dots, b_{|F|} = g$ on this path with $d_H(b_i, b_{i+1}) \geq \left\lceil \frac{|g|_H}{|F|} \right\rceil$. Then for every i there exists an $n_i \in N$ such that $d_H(b_i, n_i) \leq \text{diam}(H/N)$. As we have $|F| + 1$ elements n_i , there will be two indices $i < j$ such that n_i and n_j lie in the same coset of G . So there exists an $x \in G \cap N$ such that $n_j = x n_i$. Now we can make the following computation:

$$\begin{aligned}
|g|_H &\leq d_H(x, g) \\
&\leq d_H(x, x b_i) + d(x b_i, x n_i) + d(n_i, b_j) + d(b_j, g) \\
&\leq d_H(1, b_i) + d(b_i, g) + d(b_i, n_i) + d(n_i, b_j) \\
&\leq |g|_H - d_H(b_i, b_j) + d(b_i, n_i) + d(n_i, b_j) \\
&\leq |g|_H - \left\lceil \frac{|g|_H}{|F|} \right\rceil + 2 \text{diam}(H/N)
\end{aligned}$$

So $\frac{|g|_H}{|F|} \leq 2 \text{diam}(H/N) + 1 \leq 3 \text{diam}(H/N)$, which proves the lemma. \square

Now we can calculate the amount of intersections $N \cap \mathbb{Z}^n$ we can have such that $\text{diam}(H/N) \leq k$.

Lemma 3.7. *Let H be 2-generated and \mathbb{Z}^n -by-finite with $n \geq 3$. Then $\#\{N \cap \mathbb{Z}^n \mid N \triangleleft H, \text{diam}(H/N) \leq k\} = \mathcal{O}(k^{n^2-1})$.*

Proof. Due to Lemma 3.6 it suffices to show that $\#\{N \cap \mathbb{Z}^n \mid N \triangleleft H, \text{diam}(\mathbb{Z}^n/(N \cap \mathbb{Z}^n)) \leq k\} = \mathcal{O}(k^{n^2-1})$. So take $N \triangleleft H$ such that $\text{diam}(\mathbb{Z}^n/(N \cap \mathbb{Z}^n)) \leq k$. Then we can take $N \cap \mathbb{Z}^n$ generated by $\{x_1, \dots, x_n\}$ as in Lemma 3.4 and without loss of generality we can assume $\|x_1\| \geq \dots \geq \|x_n\|$. Now for every vector $x \in \mathbb{R}^n$ we have $d(x, \mathbb{Z}^n) \leq \frac{\sqrt{n}}{2}$, in particular we have $d(\frac{x_1}{2}, \mathbb{Z}^n) \leq \frac{\sqrt{n}}{2}$. So we can make the following computation:

$$\begin{aligned} k + \frac{\sqrt{n}}{2} &\geq \text{diam}(\mathbb{Z}^n/(N \cap \mathbb{Z}^n)) + \frac{\sqrt{n}}{2} \\ &\geq d\left(\frac{x_1}{2} + N, 0 + N\right) \\ &= \inf_{a_1, \dots, a_n \in \mathbb{Z}} \left\| \left(\frac{1}{2} + a_1 \right) x_1 + a_2 x_2 + \dots + a_n x_n \right\| \\ &\geq \frac{1}{D_n} \inf_{a_1} \left(\left| \frac{1}{2} + a_1 \right| \|x_1\| \right) && \text{by Lemma 3.4} \\ &= \frac{1}{2D_n} \|x_1\|. \end{aligned}$$

We can conclude that $2D_n k + D_n \sqrt{n} \geq \|x_1\| \geq \dots \geq \|x_n\|$. So for any i we have that x_i lies within $[-D_n(2k + \sqrt{n}), D_n(2k + \sqrt{n})]^n$.

As H is 2-generated there is an $\alpha_h \in \text{Aut}(\mathbb{Z}^n)$ different from $\pm \text{Id}$, with $\alpha_h(x) = h x h^{-1}$ where $h \in H$. Note that α_h is of finite order and note that $N \cap \mathbb{Z}^n$ is α_h -independent. So there exist a_i such that $\alpha_h(x_n) = a_1 x_1 + \dots + a_n x_n$. Note that α_h is an bounded operator on \mathbb{R}^n , which allows the following computation:

$$\begin{aligned} \|\alpha_h\| \|x_n\| &\geq \|a_1 x_1 + \dots + a_n x_n\| \\ &\geq \frac{1}{D_n} \max_i \{\|a_i x_i\|\} \\ &\geq \frac{1}{D_n} \max_i \{|a_i|\} \|x_n\|. \end{aligned}$$

So $D_n \|\alpha_h\| \geq \max_i \{|a_i|\}$.

Now we still have to count the different possibilities for N . There are fewer of these than the different possibilities for x_1, \dots, x_n , as different subgroups have different generators. Note that every possibility of x_1, \dots, x_n admits values of a_1, \dots, a_n associated to α_h .

Now we will show that for any given a sequence a_1, \dots, a_n , the number of x_1, \dots, x_n satisfying earlier conditions is bounded by $(4D_n k + 2D_n \sqrt{n} + 1)^{n^2-1}$. As the number of possibilities for any a_i is bounded by $2D_n \|\alpha_h\|$, the total number of possibilities for x_1, \dots, x_n is bounded by $(2D_n \|\alpha_h\|)^n (4D_n k + 2D_n \sqrt{n} + 1)^{n^2-1} = \mathcal{O}(k^{n^2-1})$. These earlier conditions are $D_n \|\alpha_h\| \geq \max_i \{|a_i|\}$, $2D_n k + D_n \sqrt{n} \geq \|x_1\| \geq \dots \geq \|x_n\|$ and $\alpha_h(x_n) = a_1 x_1 + \dots + a_n x_n$.

If there is an $i < n$ such that $a_i \neq 0$, then x_i can be deduced from all other x_j . So the number of possibilities of x_1, \dots, x_n is bounded by $(4D_n k + 2\sqrt{n}D_n + 1)^{(n-1)n}$.

If for every $i < n$ we have $a_i = 0$, then $a_n = \pm 1$, because otherwise α_h is not an automorphism. Since $\alpha_h \neq \pm \text{Id}$ we know that $\{x \in \mathbb{R}^n \mid \alpha_h(x) = a_n x\}$ is not the entirety of \mathbb{R}^n . Therefore it is at most an $(n-1)$ -dimensional subspace of \mathbb{R}^n , which reduces the possibilities for x_n to at most $(4D_n k + 2D_n \sqrt{n} + 1)^{n-1}$, while the possibilities of other x_1, \dots, x_{n-1} is bounded by $(4D_n k + 2D_n \sqrt{n} + 1)^{(n-1)n}$. Therefore the total number of possibilities in this case is also bounded by $(4D_n k + 2D_n \sqrt{n} + 1)^{n^2-1}$.

In conclusion we have that for any fixed sequence a_1, \dots, a_n the number of possibilities of x_1, \dots, x_n is bounded by $(4D_n k + 2D_n \sqrt{n} + 1)^{n^2-1}$. So the total number of possibilities for x_1, \dots, x_n is bounded by $(D_n \|\alpha_h\|)^n (4D_n k + 2D_n \sqrt{n} + 1)^{n^2-1}$. Therefore the possibilities of $N \cap \mathbb{Z}^n$ is bounded by that same number, which means $\#\{N \cap \mathbb{Z}^n \mid N \triangleleft H, \text{diam}(H/N) \leq k\} = \mathcal{O}(k^{n^2-1})$. \square

Now every intersection $\mathbb{Z}^n \cap N$ can be realized by multiple normal subgroups $N \triangleleft H$. However this amount is bounded. We give the following improved version of our original proposition, due to Alain Valette.

Proposition 3.8 (A.Valette). *let H be a finite group with a normal abelian subgroup A generated by $n \geq 1$ elements, and with index $d = [H : A]$. Let $S(H, A)$ be the set of normal subgroups $N \triangleleft H$ such that $N \cap A = \{1\}$. Then $|S(H, A)|$ is bounded above by a function only depending on n and d .*

Proof. Indeed let $\pi : H \rightarrow H/A$ be the quotient map. For $N_1 \in S(H, A)$, since $\pi|_{N_1}$ is injective, there are (very crudely) at most 2^d possibilities for $\pi(N_1)$.

Now we estimate how many $N_2 \in S(H, A)$ are such that $\pi(N_1) = \pi(N_2)$. The subgroup $N_1 A$ is isomorphic to the direct product $N_1 \times A$, we write its elements as pairs (n_1, a) . Now since $N_2 A = N_1 A$ we may view N_2 as the graph of a map $\alpha : N_1 \rightarrow A$ (we identify π on $N_1 \times A$ with the projection on the first factor). So we write $N_2 = \{(g, \alpha(g)) : g \in N_1\}$ and $N_1 \rightarrow N_2 : g \mapsto (g, \alpha(g))$ is an isomorphism.

Fixing $g \in N_1$, we estimate the number of possibilities for $\alpha(g)$. Since $g^d = 1$, we must have $\alpha(g)^d = 1$ in A . So we must bound d -torsion in A .

By the theory of elementary divisors, there exist integers f_1, \dots, f_k , with $f_i | f_{i+1}$, such that $A \simeq \bigoplus_{i=1}^k \mathbb{Z}/f_i \mathbb{Z}$. We have $k \leq n$ as A is n -generated. Now there are at most d elements of d -torsion in a cyclic group (by uniqueness of subgroups). So there are at most d^k elements of d -torsion in A . So the number of possibilities for $\alpha(g)$ is at most $d^k \leq d^n$.

Therefore the number of possibilities for N_2 is at most $(d^n)^{|N_2|} \leq d^{nd}$. Finally we have $|S(H, A)| \leq 2^d \cdot d^{nd}$. \square

Corollary 3.9. *Let H be \mathbb{Z}^n -by-finite for some $n \geq 3$. Then there exists a $C > 0$ such that for every $\mathcal{N} \triangleleft \mathbb{Z}^n$ of finite index the set $\#\{N \triangleleft H \mid N \cap \mathbb{Z}^n = \mathcal{N}\} \leq C$.*

This is an easy consequence of Proposition 3.8 as $\#\{N \triangleleft H \mid N \cap \mathbb{Z}^n = \mathcal{N}\} = |S(H/\mathcal{N}, \mathbb{Z}^n/\mathcal{N})|$. Now combining Lemma 3.7 and Corollary 3.9 we can control the diameter growth of H , which suffices to prove Theorem 1.4.

In the proof of Theorem 1.4 we will use a generalized version of Theorem 7 of [KV15]. We will essentially find two coarsely equivalent sequences of groups that each converge to a group in the space of marked groups. Now by combining Lemma 1.2 and Proposition 3 in [KV15] we find that these two groups are quasi-isometric.

Proof of Theorem 1.4. Suppose there is a coarse equivalence Φ between $\square_f H$ and $\square_f \mathbb{Z}^n$, with H 2-generated.

We may assume that H is residually finite, because if H is not residually finite, i.e. $\bigcap_{N \triangleleft H} N \neq \{1\}$,

then $\square_f H = \square_f H / \bigcap_{N \triangleleft H} N$ and $H / \bigcap_{N \triangleleft H} N$ is residually finite. Note that H is still 2-generated.

Now due to Lemma 1.2 there is an almost permutation ϕ between the components of $\square_f H$ and the components of $\square_f \mathbb{Z}^n$, where $\Phi|_X$ is a quasi-isometry between X and $\phi(X)$ for every component X of $\square_f H$ in the domain of ϕ . Since H is residually finite, there is a box space $\square_{(N_k)} H$ contained in $\square_f H$. Via ϕ this corresponds to a subspace $\prod_k \mathbb{Z}^n / M_k$ of $\square_f \mathbb{Z}^n$. Now this sequence

$(\mathbb{Z}^n / M_k)_k$ has a subsequence that is constant on bigger and bigger balls, i.e. there exists a sequence k_r such that $k_r \rightarrow \infty$ as $r \rightarrow \infty$ and for every $k, k' \geq k_r$ in this subsequence we have $M_k \cap B[1, r] = M_{k'} \cap B[1, r]$. Now due to a generalized version of Theorem 7 of [KV15] H is quasi-isometric a quotient of \mathbb{Z}^n , because the intersection of the subsequence M_k converges to a normal subgroup of \mathbb{Z}^n . So H is virtually \mathbb{Z}^m with $m \leq n$, due to the quasi-isometric rigidity of \mathbb{Z}^m .

Due to Lemma 3.7 we have $\#\{N \cap \mathbb{Z}^n \mid N \triangleleft H, \text{diam}(H/N) \leq k\} = \mathcal{O}(k^{m^2-1})$ and due to Corollary 3.9 we have $\#\{N \triangleleft H \mid \text{diam}(H/N) \leq k\} = \mathcal{O}(k^{m^2-1})$. However due to Proposition 3.1 we have that $\#\{N \triangleleft \mathbb{Z}^n \mid \text{diam}(\mathbb{Z}^n/N) \leq k\} = \mathcal{O}(k^{m^2-1})$, but as $m \leq n$ this is in contradiction with Proposition 3.2. \square

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